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Genera defined by hyperelliptic integrals and Siegel modular functions

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Abstract

We study genera defined by hyperelliptic integrals. The associated formal group laws select a particular set of rational generators of the complex cobordism ring. Hyperelliptic genera and their kernels are concisely described in terms of these generators. Using Thomae's formula which expresses branch points of hyperelliptic curves in terms of hyperelliptic theta constants, we express values of hyperelliptic genera in terms of hyperelliptic theta constants evaluated at period matrices of the associated hyperelliptic curves. In particular, in genus 2 case, we obtain a hyperelliptic genus whose values lie in the ring of level 2 genus 2 Siegel modular functions.

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1. Introduction and summary of results

In [16], the notion of elliptic genus was introduced. This is a ring map from the complex cobordism ring $\varphi_{\mathrm{ell}}: \Omega_*^{\mathrm{U}} \to \mathbb{Z}[\frac{1}{2}][\delta, \varepsilon]$ into a polynomial algebra generated by δ and ε over $\mathbb{Z}[\frac{1}{2}]$. For general information on genera, see [7,17]. The name "elliptic" comes from a fact that its logarithm [1,2]

$$\log_\varphi(X) = \sum_{n \geqslant 0} \frac{\varphi_{\mathrm{ell}}([\mathbb{C}P^n])}{n+1} X^{n+1}$$

is given by a formal elliptic integral [3,22] of the form

$$\log_\varphi(X) = \int_0^X \frac{\mathrm{d}t}{\sqrt{1 - 2\delta t^2 + \varepsilon t^4}}, \tag{1.1}$$

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where X is a formal variable. The right-hand side of (1.1) should be understood as a formal power series in X. This integral originally appeared in the study of circle actions on Spin manifolds [16,14,9], where the use of elliptic functions played an interesting role. For geometric aspects of elliptic genera and further topics, see [8,13,18,23,24].

In this paper, we study the behavior of a genus whose logarithm is defined by a hyperelliptic integral of the form

$$\int_0^X \frac{P(t)\,\mathrm{d}t}{\sqrt{1-2\theta_1 t+\theta_2 t^2+\sum_{k=3}^{n}\{12/[k(k-1)]\}\theta_k t^k}}. \tag{1.2}$$

Here $P(t)$ is a polynomial whose constant term is nonzero. For the origin of the rational coefficients in the above expression, see (2.11) and theorem below. We call the corresponding genus a *hyperelliptic genus* [20]. The name comes from hyperelliptic curves: if all the above θ's are complex numbers, and the equation

$$F(X)=1-2\theta_1 X+\theta_2 X^2+\sum_{k=3}^{n}\frac{12}{k(k-1)}\theta_k X^k=0$$

has distinct roots, then the curve Σ defined over $\mathbb{C}$ by the equation $Y^2=F(X)$ is a smooth hyperelliptic curve of genus $g=[(n-1)/2]$, that is, it admits a morphism of degree 2 onto the Riemann sphere $\mathbb{C}P^1$. If $P(X)$ is a polynomial of degree at most $g-1$, then the integrand in (1.2) represents a global holomorphic differential form ω on Σ and any global holomorphic form on Σ is of this form.

For the description of the behavior of a hyperelliptic genus, the following particular choice of rational generators $\{x_i\}_{i\geqslant 1}$ of the complex cobordism ring Ω_*^{U} turns out to be useful:

$$\begin{aligned}
&x_1=[\mathbb{C}P^1],\quad x_2=3[\mathbb{C}P^1]^2-2[\mathbb{C}P^2],\\
&x_{k+2}=[H_{k,3}]-[\mathbb{C}P^3][\mathbb{C}P^{k-1}]-2[\mathbb{C}P^1][H_{k,2}]+2[\mathbb{C}P^1][\mathbb{C}P^2][\mathbb{C}P^{k-1}],\\
&\qquad k\geqslant 1.
\end{aligned} \tag{1.3}$$

Here $H_{ij}\subset\mathbb{C}P^i\times\mathbb{C}P^j$ is a Milnor manifold, namely a hypersurface of degree $(1,1)$. Note that the above rational generators are in fact integral elements in the ring Ω_*^{U}. The above elements $\{x_i\}$ are naturally chosen by hyperelliptic integrals.

Our first main result is the following description of the hyperelliptic genera when $P(t)\equiv 1$. Let $\theta_1,\theta_2,\ldots,\theta_n$ be indeterminates.

Theorem. *Let $\varphi:\Omega_*^{\mathrm{U}}\to\mathbb{Q}[\theta_1,\theta_2,\ldots,\theta_n]$ be the hyperelliptic genus associated to a logarithm given by*

$$\log_\varphi(X)=\int_0^X \frac{\mathrm{d}t}{\sqrt{1-2\theta_1 t+\theta_2 t^2+\sum_{k=3}^{n}\{12/[k(k-1)]\}\theta_k t^k}}. \tag{1.4}$$

Then, in terms of the rational generators $\{x_i\}_{i\geqslant 1}$, *the genus* φ *is described by*

$$\varphi(x_\ell)=\theta_\ell \quad \textit{for } 1\leqslant \ell \leqslant n,$$
$$\varphi(x_\ell)=0 \quad \textit{for } \ell > n. \tag{1.5}$$

The kernel $\operatorname{Ker}\varphi$ *is the ideal* $(x_{n+1},x_{n+2},\ldots)_{\mathbb{Q}}\cap\Omega_*^{\mathrm{U}}$, *where* $(\ )_{\mathbb{Q}}$ *means an ideal in the ring* $\Omega_*^{\mathrm{U}}\otimes\mathbb{Q}$.

Note that it takes the entire family of hyperelliptic genera to single out the above set of generators.

Hyperelliptic genera can be used to construct a sequence of multiplicative idempotents on $\Omega_*^{\mathrm{U}}\otimes\mathbb{Q}$ (Corollary 2.3). For any finite set $I\subset\{1,2,\ldots,n,\ldots\}$, there exists an idempotent hyperelliptic genus $\varphi:\Omega_*^{\mathrm{U}}\otimes\mathbb{Q}\to\Omega_*^{\mathrm{U}}\otimes\mathbb{Q}$ such that

$$\varphi(x_i)=x_i \quad \text{for } i\in I,$$
$$\varphi(x_j)=0 \quad \text{for } j\notin I. \tag{1.6}$$

We can also construct hyperelliptic genera from the oriented cobordism ring (Corollary 2.4).

We can give a similar description of a genus whose logarithm is of a more general form (1.2). See Proposition 2.5.

Our method to analyze hyperelliptic genera can be applied to study genera whose logarithm is given by integrals of the form

$$\int_0^X \frac{P(t)\,\mathrm{d}t}{\sqrt[m]{F(t)}}, \tag{1.7}$$

where $P(t)$ and $F(t)$ are polynomials whose leading terms are 1. This can be carried out with a little more calculation for the case $m=3$. However, for larger m, the calculation becomes increasingly complicated.

One of the main features of elliptic genera is that the elements δ,ε in (1.1) can be interpreted as modular functions (of weight 0) in the upper half plane for a level 2 congruence subgroup. Theta functions and Jacobi forms [4] are useful tools in this context. There are several possible formulae [19, Section 1.4, Theorem 4]. For example,

$$\delta=-\frac{1}{8}q^{(-1/4)}\left\{\prod_{\ell\geqslant 1}\left(\frac{1+q^{\ell-1/2}}{1+q^{\ell}}\right)^4+16q^{1/2}\prod_{\ell\geqslant 1}\left(\frac{1+q^{\ell}}{1+q^{\ell-1/2}}\right)^4\right\},\quad \varepsilon=1. \tag{1.8}$$

One can also express δ and ε as level 2 modular forms of weight 2 and 4 [25]:

$$\delta=-\frac{1}{8}-3\sum_{n\geqslant 1}\left(\sum_{\substack{d\mid n\\ d:\ \text{odd}}} d\right)q^n,\quad \varepsilon=\sum_{n\geqslant 1}\left(\sum_{\substack{d\mid n\\ n/d:\ \text{odd}}} d^3\right)q^n. \tag{1.9}$$

Along a similar line, we show that the coefficients θ_k in (1.2) can be expressed as level 2 Siegel modular functions (see Definition 3.1) on the space of hyperelliptic period matrices, which is a subspace of the Siegel upper half space. For the description of the result, see Propositions 4.1 and 4.2. This is our second result. These Siegel modular functions can be expressed as quotients of genus 2 theta constants of half integral characteristics (see (3.6) and (3.7)). We give explicit formulae for these Siegel modular forms in Proposition 3.3.

2. Algebraic hyperelliptic genera

We recall that the universal formal group law $F_{\mathrm{MU}}(X,Y)$, which is defined over the complex cobordism ring Ω_*^{U}, and its logarithm $\log_{\mathrm{MU}}(X)$, which is also denoted by $g_{\mathrm{MU}}(X)$ and which is defined over $\Omega_*^{\mathrm{U}} \otimes \mathbb{Q}$, are power series in indeterminates X, Y given by

$$g_{\mathrm{MU}}(X) = \sum_{n \geqslant 0} \frac{[\mathbb{C}P^n]}{n+1} X^{n+1},$$

$$F_{\mathrm{MU}}(X,Y) = g_{\mathrm{MU}}^{-1}(g_{\mathrm{MU}}(X) + g_{\mathrm{MU}}(Y)). \tag{2.1}$$

Note that the formal derivative $g'_{\mathrm{MU}}(X)$ with respect to X has coefficients in Ω_*^{U}.

Now we study the genus $\varphi : \Omega_*^{\mathrm{U}} \to \mathbb{Q}[\theta_1, \theta_2, \ldots, \theta_n]$ whose logarithm $g_\varphi(X)$ is given by

$$g_\varphi(X) = \int_0^X \frac{\mathrm{d}t}{\sqrt{1 - 2\theta_1 t + \theta_2 t^2 + \sum_{k=3}^{n} \{12/[k(k-1)]\}\theta_k t^k}}. \tag{2.2}$$

We see at once that

$$\frac{1}{g'_\varphi(X)^2} = 1 - 2\theta_1 X + \theta_2 X^2 + \sum_{k=3}^{n} \frac{12}{k(k-1)} \theta_k X^k. \tag{2.2$'$}$$

Using the derivative $\log'_{\mathrm{MU}}(X) = g'_{\mathrm{MU}}(X)$ of the logarithm of the universal formal group law, we define elements $\lambda_n \in \Omega_{2n}^{\mathrm{U}}$ for $n \geqslant 1$ by

$$\frac{1}{g'_{\mathrm{MU}}(X)^2} = 1 + \sum_{n \geqslant 1} \lambda_n X^n \in \Omega_*^{\mathrm{U}}[\![X]\!], \tag{2.3}$$

where $g_{\mathrm{MU}}(X)$ is as in (2.1). By universality of $F_{\mathrm{MU}}(X,Y)$, comparison of (2.2$'$) and (2.3) shows that images of λ_n under φ are coefficients of the degree n polynomial on the right-hand side of (2.2$'$). That is,

$$\varphi(\lambda_1) = -2\theta_1, \quad \varphi(\lambda_2) = \theta_2, \quad \varphi(\lambda_k) = \frac{12}{k(k-1)}\theta_k, \quad 3 \leqslant k \leqslant n,$$

$$\varphi(\lambda_k) = 0, \quad k > n. \tag{2.4}$$

Thus, giving a simple description of the genus φ reduces to finding a simple description of elements λ_n's. Of course, it is straightforward to see that (2.1) and (2.3) give the following relation on λ_n's:

$$\left(1+\sum_{n\geqslant 1}\lambda_n X^n\right)\left(1+\sum_{n\geqslant 1}[\mathbb{C}P^n]X^n\right)^2=1. \tag{2.5}$$

From this, elements λ_n can be expressed as polynomials in $[\mathbb{C}P^k]$s. But these polynomials are rather too complicated. Fortunately, a much simpler description of λ_n's exists in terms of Milnor manifolds H_{ij} for $i,j\geqslant 0$, $i+j\geqslant 2$. The manifold H_{ij} is a degree $(1,1)$ hypersurface in $\mathbb{C}P^i\times\mathbb{C}P^j$. Note that $H_{i+1,0}=H_{0,i+1}=\mathbb{C}P^i$.

We define a two variable power series $H(u,v)\in\Omega_*^{\mathrm{U}}[\![u,v]\!]$ in terms of Milnor manifolds by

$$H(u,v)=u+v+\sum_{\substack{i,j\geqslant 0\\ i+j\geqslant 2}}[H_{ij}]u^iv^j. \tag{2.6}$$

The following identity is well known [1, Proposition 10.6]:

$$H(u,v)=g'_{\mathrm{MU}}(u)g'_{\mathrm{MU}}(v)F_{\mathrm{MU}}(u,v). \tag{2.7}$$

It is also well known that Milnor manifolds H_{ij} generate the entire complex cobordism ring Ω_*^{U}. We use the identity (2.7) to obtain a simple expression of λ_n's in terms of Milnor manifolds. To simplify the notation, let

$$h_{ij}=[H_{ij}],\quad p_k=[\mathbb{C}P^k]\quad \text{for } i,j,k\geqslant 0,\ i+j\geqslant 2,$$

where $p_0=1$. By comparing coefficients of v^i for $1\leqslant i\leqslant 3$ in the identity (2.7), and omitting MU from our notations g_{MU} and $F_{\mathrm{MU}}(u,v)$, we obtain

$$\begin{aligned}
1+\sum_{i\geqslant 1}h_{i1}u^i&=p_1ug'(u)+g'(u)\frac{\partial F}{\partial v}(u,0),\\
\sum_{i\geqslant 0}h_{i2}u^i&=p_2ug'(u)+p_1g'(u)\frac{\partial F}{\partial v}(u,0)+\frac{g'(u)}{2}\frac{\partial^2F}{\partial v^2}(u,0),\\
\sum_{i\geqslant 0}h_{i3}u^i&=p_3ug'(u)+p_2g'(u)\frac{\partial F}{\partial v}(u,0)+p_1\frac{g'(u)}{2}\frac{\partial^2F}{\partial v^2}(u,0)+\frac{g'(u)}{3!}\frac{\partial^3F}{\partial v^3}(u,0).
\end{aligned} \tag{2.8}$$

In view of (2.3), we let $b(u)=1/g'_{\mathrm{MU}}(u)\in\Omega_*^{\mathrm{U}}[\![u]\!]$. We want to obtain a simple expression of $b(u)^2$.

Since $g_{\mathrm{MU}}(F_{\mathrm{MU}}(u,v))=g_{\mathrm{MU}}(u)+g_{\mathrm{MU}}(v)$, differentiating this identity by the variable v several times and using the identities

$$b'(u)=-\frac{g''(u)}{g'(u)^2},\quad \frac{1}{2}\{b(u)^2\}''=-\frac{g'''(u)}{g'(u)^3}+3\left(\frac{g''(u)}{g'(u)^2}\right)^2,$$

we obtain the following identities:

$$g'(u)\frac{\partial F}{\partial v}(u,0)=1,\qquad \frac{g'(u)}{2}\frac{\partial^2 F}{\partial v^2}(u,0)=\frac{1}{2}\{p_1+b'(u)\},$$

$$\frac{g'(u)}{3!}\frac{\partial^3 F}{\partial v^3}(u,0)=\frac{p_2}{3}+\frac{p_1}{2}b'(u)+\frac{1}{12}\{b(u)^2\}''. \tag{2.9}$$

Of course, if we examine coefficients of higher powers of v, we get information on $\{b(u)^m\}^{(m)}$, the mth derivative of the mth power of $b(u)$, for any m. This information can be used to study genera defined by logarithm of the form (1.7). Here we concentrate on the hyperelliptic case $m=2$.

Combining (2.8) and (2.9), we easily obtain the next lemma.

Lemma 2.1. *Let* $b(u)=1/g'_{\mathrm{MU}}(u)\in\Omega_*^{\mathrm{U}}[\![u]\!]$. *Then we have*

$$\frac{b'(u)+p_1}{2}=\sum_{i\geqslant 1}(h_{i,2}-p_2p_{i-1})u^i,$$

$$\frac{\{b(u)^2\}''+(4p_2-6p_1^2)}{12}=\sum_{i\geqslant 1}(h_{i,3}-2p_1h_{i,2}-p_3p_{i-1}+2p_1p_2p_{i-1})u^i. \tag{2.10}$$

By integrating the first identity in (2.10), we obtain the following simple expression of $1/g'_{\mathrm{MU}}(u)$ in terms of Milnor manifolds:

$$\frac{1}{g'_{\mathrm{MU}}(u)}=b(u)=1-p_1u+\sum_{i\geqslant 1}\frac{2}{i+1}(h_{i,2}-p_2p_{i-1})u^{i+1}.$$

We remark that elements $\{x_k\}_{k\geqslant 1}$ in Ω_*^{U} defined in (1.3) come from the right-hand side of the second identity of (2.10). By integrating it twice, we get

$$\frac{1}{\{g'_{\mathrm{MU}}(u)\}^2}=b(u)^2=1-2x_1u+x_2u^2+\sum_{k\geqslant 3}\frac{12}{k(k-1)}x_ku^k. \tag{2.11}$$

The universality of $F_{\mathrm{MU}}(u,v)$ yields the following description of the hyperelliptic genus.

Theorem 2.2 (Hyperelliptic genera)**.** *Let* $\theta_1,\theta_2,\ldots,\theta_n$ *be indeterminates. Let*

$$\varphi:\Omega_*^{\mathrm{U}}\to\mathbb{Q}[\theta_1,\theta_2,\ldots,\theta_n] \tag{2.12}$$

be the hyperelliptic genera whose logarithm $g_\varphi(X)$ *is given by*

$$g_\varphi(X)=\int_0^X\frac{\mathrm{d}t}{\sqrt{1-2\theta_1t+\theta_2t^2+\sum_{k=3}^{n}\{12/[k(k-1)]\}\theta_kt^k}}. \tag{2.13}$$

Then in terms of the rational generators $\{x_k\}_{k\geqslant 1}$ *in* (1.3) *the genus* φ *is described as follows*:

$$\varphi(x_k)=\begin{cases}\theta_k & \text{for } 1\leqslant k\leqslant n,\\ 0 & \text{for } k>n.\end{cases} \tag{2.14}$$

In particular, *the kernel of the genus* φ *is given by the ideal*

$$\mathrm{Ker}\,\varphi=(x_{n+1},x_{n+2},\ldots)_{\mathbb{Q}}\cap\Omega_*^{\mathrm{U}}. \tag{2.15}$$

Here, $(\cdots)_{\mathbb{Q}}$ *means the ideal in the ring* $\Omega_*^{\mathrm{U}}\otimes\mathbb{Q}$.

Proof. Formulae (2.14) follow from (2.2′), (2.4), and (2.11), using the universality of the formal group law over Ω_*^{U}. The description of the kernel of the genus φ follows from this. □

Note that the family of hyperelliptic genera of the form described in Theorem 2.2 with various k canonically selects a set of rational generators $\{x_k\}_{k\geqslant 1}$ of the complex cobordism ring. It also selects rational generators of oriented cobordism ring (see Corollary 2.4).

Remark. In Theorem 2.2, if we assign degree to elements θ_k's by letting $\deg\theta_k=2k$, then we have a degree preserving genus φ. Later in Propositions 4.1 and 4.2, we will express θ_k's as Siegel modular functions (of weight 0) on Siegel upper half spaces. In this case, there is no notion of degree for θ_k's.

By specializing θ_k's, we can obtain interesting hyperelliptic genera. For example, we can construct a sequence of multiplicative idempotents on the ring $\Omega_*^{\mathrm{U}}\otimes\mathbb{Q}$. The proof of the next corollary is straightforward from Theorem 2.2.

Corollary 2.3 (Hyperelliptic idempotents). *For each* $n\geqslant 0$, *let* $\varphi_n:\Omega_*^{\mathrm{U}}\to\Omega_*^{\mathrm{U}}\otimes\mathbb{Q}$ *be a hyperelliptic genus defined by a logarithm*

$$g_n(X)=\int_0^X\frac{\mathrm{d}t}{\sqrt{1-2x_1t+x_2t^2+\sum_{k=3}^{n}\{12/[k(k-1)]\}x_kt^k}}. \tag{2.16}$$

Then the ring map $\varphi_n:\Omega_*^{\mathrm{U}}\otimes\mathbb{Q}\to\Omega_*^{\mathrm{U}}\otimes\mathbb{Q}$ *is a multiplicative idempotent whose image is* $\mathbb{Q}[x_1,\ldots,x_n]$.

Thus, *the collection of idempotents* $\{\varphi_n\}_{n\geqslant 1}$ *exhausts the entire ring* $\Omega_*^{\mathrm{U}}\otimes\mathbb{Q}$.

When the logarithm $g(X)$ is an odd power series, the corresponding genus factors through Ω_*^{SO}. In the hyperelliptic context, we have the following corollary.

Corollary 2.4. *Let* $\gamma_1,\ldots,\gamma_n$ *be indeterminates. Let*

$$\varphi:\Omega_*^{\mathrm{U}}\to\mathbb{Q}[\gamma_1,\gamma_2,\ldots,\gamma_n] \tag{2.17}$$

be a hyperelliptic genus whose logarithm is given by

$$g_{\varphi}(X)=\int_0^X \frac{\mathrm{d}t}{\sqrt{1-2\gamma_1 t^2+\sum_{k=2}^{n}\{6/[k(2k-1)]\}\gamma_k t^{2k}}}. \tag{2.18}$$

Then φ is described by

$$\varphi([\mathbb{C}P^{2k+1}])=0 \quad \textit{for all } k \geqslant 0,$$

$$\varphi([\mathbb{C}P^2])=\gamma_1, \quad \varphi([H_{2k-2,3}])=\begin{cases}\gamma_k & \textit{if } 2\leqslant k\leqslant n,\\ 0 & \textit{if } k>n.\end{cases} \tag{2.19}$$

Thus the genus φ factors through Ω_*^{SO} and gives

$$\varphi:\Omega_*^{\mathrm{SO}}\to\mathbb{Q}[\gamma_1,\gamma_2,\ldots,\gamma_n].$$

Proof. Since logarithm (2.18) is an odd power series, we have $\varphi([\mathbb{C}P^{2k+1}])=0$ for all k. Thus, φ kills all complex odd-dimensional elements in Ω_*^{U}. Then, Theorem 2.2 says that

$$\varphi(x_{\mathrm{odd}})=0, \quad \varphi(x_2)=-2\gamma_1, \quad \varphi(x_{2k})=\begin{cases}\gamma_k & \text{for } 2\leqslant k\leqslant n,\\ 0 & \text{if } k>n.\end{cases}$$

Applying φ to (1.3) and noting that all $[\mathbb{C}P^{\mathrm{odd}}]$ are killed by φ, we have

$$\varphi(x_2)=-2\varphi([\mathbb{C}P^2]), \quad \varphi(x_{2k})=\varphi([H_{2k-2,3}]).$$

This proves (2.19). Since $[\mathbb{C}P^{\mathrm{odd}}]$, $[\mathbb{C}P^2]$, and $[H_{2k-2,3}]$ with $k\geqslant 2$ generate Ω_*^{U} rationally, (2.19) completely describe the genus φ. □

Next we discuss general hyperelliptic genus whose logarithm is of the form (1.2). Let Λ be a $\mathbb{Q}$-algebra and let $P(X)$ be a polynomial in an indeterminate X with coefficients in Λ with the leading term 1. Let its square be given by

$$P(X)^2=1+\sum_{i\geqslant 1}c_iX^i\in\Lambda[\![X]\!]. \tag{2.20}$$

We define elements $x_k^{[P]}\in\Omega_*^{\mathrm{U}}\otimes\Lambda$ for $k\geqslant 1$ by the following identity (compare with (2.11)):

$$\frac{P(X)^2}{(\sum_{n\geqslant 0}[\mathbb{C}P^n]X^n)^2}=1-2x_1^{[P]}X+x_2^{[P]}X^2+\sum_{k\geqslant 3}\frac{12}{k(k-1)}x_k^{[P]}X^k. \tag{2.21}$$

More explicitly, elements $x_k^{[P]}$ are given by

$$x_1^{[P]}=x_1-\tfrac{1}{2}c_1, \quad x_2^{[P]}=x_2-2c_1x_1+c_2,$$

$$x_n^{[P]}=\sum_{k=3}^{n}\frac{n(n-1)}{k(k-1)}x_kc_{n-k}+\frac{n(n-1)}{12}(c_{n-2}x_2-2c_{n-1}x_1+c_n), \quad n\geqslant 3. \tag{2.22}$$

With these notations, our description of general hyperelliptic genus goes as follows.

Proposition 2.5 (General hyperelliptic genera). *Let Λ be a $\mathbb{Q}$-algebra. Let*

$$\varphi^{[P]} : \Omega_*^{\mathrm{U}} \otimes \Lambda \to \Lambda[\theta_1, \theta_2, \ldots, \theta_n] \tag{2.23}$$

be a general Λ-linear hyperelliptic genus defined by a logarithm

$$g(X) = \int_0^X \frac{P(t)\,\mathrm{d}t}{\sqrt{1 - 2\theta_1 t + \theta_2 t^2 + \sum_{k=3}^{n} \{12/[k(k-1)]\}\theta_k t^k}}, \tag{2.24}$$

where $P(t)$ is a power series in t with coefficients in Λ whose square is given by (2.20). *Then the genus $\varphi^{[P]}$ is described by*

$$\varphi^{[P]}(x_K^{[P]}) = \begin{cases} \theta_k & \text{if } 1 \leqslant k \leqslant n, \\ 0 & \text{if } k > n. \end{cases} \tag{2.25}$$

The kernel of $\varphi^{[P]}$ is given by the ideal $(x_{n+1}^{[P]}, x_{n+2}^{[P]}, \ldots) \subset \Omega_*^{\mathrm{U}} \otimes \Lambda$.

Proof. From (2.24), we have

$$\frac{P(X)^2}{g'(X)^2} = 1 - 2\theta_1 t + \theta_2 t^2 + \sum_{k=3}^{n} \frac{12}{k(k-1)}\theta_k t^k.$$

Since the left-hand side is the image under $\varphi^{[P]}$ of $P(X)^2/(\sum_{n \geqslant 0} [\mathbb{C}P^n] X^n)^2$, comparing with (2.21), we have the description (2.25). □

3. Theta functions and roots of algebraic equations: Thomae's formula

In this section, we briefly describe materials on theta functions in several variables. A basic reference is [15]. Thomae's formula is then used to express branch points of hyperelliptic curves in terms of hyperelliptic theta constants. In the next section, theta functions are used to construct analytic hyperelliptic genera with values in the ring of Siegel modular functions. For general background on Riemann surfaces, see [5,6].

Let g be a positive integer. The Siegel upper half space is, by definition, the set of all complex symmetric matrices with positive definite imaginary part:

$$\mathfrak{H}_g = \{\Omega \in M_g(\mathbb{C}) \mid {}^t\Omega = \Omega, \operatorname{Im} \Omega > 0\}. \tag{3.1}$$

This is an open subset of $\mathbb{C}^{g(g+1)/2}$. Let $\mathrm{Sp}(2g, \mathbb{Z})$ be the symplectic modular group defined by

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2g}(\mathbb{Z}) \,\middle|\, \begin{pmatrix} {}^tA & {}^tC \\ {}^tB & {}^tD \end{pmatrix} \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix} \right\}. \tag{3.2}$$

The symplectic modular group acts on the Siegel upper half space by

$$\begin{aligned} &\mathrm{Sp}(2g, \mathbb{Z}) \times \mathfrak{H}_g \to \mathfrak{H}_g, \\ &\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \Omega \right) \mapsto (A\Omega + B)(C\Omega + D)^{-1}. \end{aligned} \tag{3.3}$$

Let $\Gamma_2 \subset \mathrm{Sp}(2g,\mathbb{Z})$ be the level 2 subgroup defined by

$$\Gamma_2 = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g,\mathbb{Z}) \,\middle|\, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv I_{2g} \bmod 2 \right\}. \tag{3.4}$$

The theta function $\vartheta : \mathbb{C}^g \times \mathfrak{H}_g \to \mathbb{C}$ is a holomorphic function defined by

$$\vartheta(\vec{z},\Omega) = \sum_{\vec{n}\in\mathbb{Z}^g} \exp(\pi i \vec{n}\Omega^t\vec{n} + 2\pi i \vec{n}\cdot{}^t\vec{z}), \tag{3.5}$$

where $\vec{z}$ and $\vec{n}$ are regarded as row vectors. The above sum converges absolutely and uniformly in a suitable sense and hence $\vartheta(\vec{z},\Omega)$ is a holomorphic function. The theta function is quasi-periodic on the lattice $L_\Omega = \mathbb{Z}^g + \Omega\mathbb{Z}^g$:

$$\begin{aligned} &\vartheta(\vec{z}+\vec{m},\Omega) = \vartheta(\vec{z},\Omega), \\ &\vartheta(\vec{z}+\Omega\vec{m},\Omega) = \exp(-\pi i \vec{m}\Omega^t\vec{m} - 2\pi i \vec{m}^t\vec{z})\vartheta(\vec{z},\Omega), \end{aligned} \tag{3.6}$$

for any $\vec{m}\in\mathbb{Z}^g$. We can also define theta functions with half integral characteristics:

$$\vartheta\begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix}(\vec{z},\Omega) = \sum_{\vec{n}\in\mathbb{Z}^g} \exp(\pi i(\vec{n}+\vec{a})\Omega^t(\vec{n}+\vec{a}) + 2\pi i(\vec{n}+\vec{a})^t(\vec{n}+\vec{b})), \tag{3.7}$$

where $\vec{a},\vec{b}\in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g$ are row vectors whose entries are 0 or $\frac{1}{2}$. A given characteristic is called *even* if $4\vec{a}\cdot{}^t\vec{b}$ is even, and *odd* if $4\vec{a}\cdot{}^t\vec{b}$ is odd. The value of the above theta function at $\vec{z}=\vec{0}$ is called *theta constants*. Just like one variable theta functions, they have modularity properties.

Definition 3.1 (*Siegel modular form*). Let $\Gamma \subset \mathrm{Sp}(2g,\mathbb{Z})$ be a subgroup of finite index. A holomorphic function f on the Siegel upper half space $\mathfrak{H}_g$ is called a Siegel modular form of weight k and of level Γ if for all

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma,$$

f satisfies

$$f((A\Omega+B)(C\Omega+D)^{-1}) = \det(C\Omega+D)^k f(\Omega). \tag{3.8}$$

The following fact is well known [15]. (It is not explicitly proved there, but it can be proved easily using results in [15]: just use (5.2), proof of (5.1) on p. 194, (5.9), and Proposition A3.)

Proposition 3.2. *For any $\vec{a},\vec{b}\in\frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g$, the function*

$$\vartheta\begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix}(0,\Omega)^4$$

on $\mathfrak{H}_g$ is a Siegel modular form of weight 2 and of level Γ_2.

We are primarily interested in matrices Ω which appear as period matrices of hyperelliptic curves. Given a smooth hyperelliptic curve Σ: $Y^2 = F(X)$ of genus g with ordered branch points on $\mathbb{C}P^1$, there is a traditional way to choose a symplectic basis $\{A_i, B_j\}_{i,j=1}^{g}$ of $H_1(\Sigma, \mathbb{Z})$ after choosing a simple closed curve on $\mathbb{C}P^1$ which passes through branch points in a given order [15, IIIa, Section 5]. If we use a different simple closed curve, then the corresponding symplectic bases change by the action of an element in $\Gamma_2 \subset \mathrm{Sp}(2g, \mathbb{Z})$, and every element in Γ_2 occurs in this way [15, IIIa, Lemma 8.12].

Since the space of holomorphic differential forms on Σ is g-dimensional, we choose its basis $\{\omega_i\}$ by requiring $\int_{A_i} \omega_j = \delta_{ij}$ for $1 \leqslant i,\ j \leqslant g$. The period matrix $\Omega \in \mathfrak{H}_g$ for Σ is then defined by

$$\Omega_{ij} = \int_{B_i} \omega_j. \tag{3.9}$$

When two symplectic bases of $H_1(\Sigma; \mathbb{Z})$ are related by $\gamma \in \mathrm{Sp}(2g, \mathbb{Z})$, the corresponding period matrices are related by the action of γ on $\mathfrak{H}_g$. The set of hyperelliptic period matrices of genus g form a $2g-1$ dimensional subset of $\mathfrak{H}_g$. Recall that $\mathfrak{H}_g$ is $g(g+1)/2$ dimensional.

It is a remarkable fact that for a hyperelliptic curve Σ, hyperelliptic theta constants are directly related to cross ratios of branch points. This follows from Thomae's formula. To describe the formula, let $B = \{x_1, x_2, \ldots, x_{2g+1}, \infty\}$ be the set of all the ordered branch points of Σ. As before, a choice of simple closed curve through branch points provides us with a symplectic basis and a period matrix Ω. Let $B' = \{1, 2, \ldots, 2g+1\}$ be the index set for finite branch points. Let $U = \{1, 3, \ldots, 2g+1\}$ be the set of odd indices. For any subset $S \subset B'$, let $S \circ U = S \cup U - S \cap U$, the symmetric difference. For $1 \leqslant k \leqslant 2g$, let η_k be a $2 \times g$ matrix given by

$$\eta_{2i-1} = \begin{pmatrix} 0 & \ldots & 0 & \frac{1}{2} & 0 & \ldots & 0 \\ \frac{1}{2} & \ldots & \frac{1}{2} & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad \eta_{2i} = \begin{pmatrix} 0 & \ldots & 0 & \frac{1}{2} & 0 & \ldots & 0 \\ \frac{1}{2} & \ldots & \frac{1}{2} & \frac{1}{2} & 0 & \ldots & 0 \end{pmatrix}, \tag{3.10}$$

for $1 \leqslant i \leqslant g$, and the last nontrivial column is the ith column. We let

$$\eta_S = \sum_{k \in S} \eta_k \quad \text{for } S \subset B'.$$

Using the notations as above, we can state Thomae's formula [15, IIIa, Section 8].

Thomae's Formula. *There exists a constant c such that for all $S \subset B'$ with $\#S$ even, we have*

$$\vartheta[\eta_S](\Omega)^4 = \begin{cases} 0 & \text{if } \#(S \circ U) \neq g+1, \\ c(-1)^{\#(S\cap U)} \prod (x_i - x_j)^{-1} & \text{if } \#(S \circ U) = g+1. \end{cases} \tag{3.11}$$

where in the second case, the product runs over all i, j such that $i \in (S \circ U)$ and $j \in B' - (S \circ U)$.

Note that for any subset $V \subset B'$, the set $S = V \circ U$ is such that $S \circ U = V$. Hence if $\#V = g+1$, then the set $S = V \circ U$ has the property $\#(S \circ U) = \#V = g+1$ and $\#S$ is even, which is the interesting case in (3.11).

Let $k, \ell, m \in B'$, and choose a partition B' such that $B' = W_1 \amalg W_2 \amalg \{k, \ell, m\}$, where $\#W_1 = \#W_2 = g-1$. Then from Thomae's formula, we easily get

$$\frac{(x_k - x_\ell)^2}{(x_k - x_m)^2} = \frac{\vartheta[\eta_{(W_1+\ell+k)\circ U}](\Omega)^4 \cdot \vartheta[\eta_{(W_2+\ell+k)\circ U}](\Omega)^4}{\vartheta[\eta_{(W_1+m+k)\circ U}](\Omega)^4 \cdot \vartheta[\eta_{(W_2+m+k)\circ U}](\Omega)^4}. \tag{3.12}$$

Note that in this expression, it does not matter which W_1 and W_2 we choose. For genus $g = 2$ case, there is no ambiguity in the choice of W_1 and W_2 in the above formula for a given $\{k, \ell, m\} \subset \{1, 2, 3, 4, 5\}$. Using (3.12) and the identity

$$1 + \frac{(x_k - x_\ell)^2}{(x_k - x_m)^2} - \frac{(x_m - x_\ell)^2}{(x_m - x_k)^2} = 2\frac{(x_k - x_\ell)}{(x_k - x_m)},$$

we obtain the following formula:

$$\frac{x_k - x_\ell}{x_k - x_m} = \tfrac{1}{2}\{1 + \theta_{km}^{k\ell}(\Omega)\}, \tag{3.13}$$

where

$$\theta_{km}^{k\ell}(\Omega) = \frac{\vartheta[\eta_{(W_1+k+\ell)\circ U}]^4 \cdot \vartheta[\eta_{(W_2+k+\ell)\circ U}]^4 - \vartheta[\eta_{(W_1+m+\ell)\circ U}]^4 \cdot \vartheta[\eta_{(W_1+m+\ell)\circ U}]^4}{\vartheta[\eta_{(W_1+k+m)\circ U}]^4 \cdot \vartheta[\eta_{(W_1+k+m)\circ U}]^4}.$$

By the action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathbb{C}P^1$, we can assume that $x_1 = 0$, $x_2 = 1$. So the set of branch points are $\{0, 1, x_3, \ldots, x_{2g+1}, \infty\}$. Let $W_1^{(\ell)}, W_2^{(\ell)} \subset B'$ be any subset such that $W_1 \amalg W_2 \amalg \{1, 2, \ell\} = B'$. Then, we have the following formulae.

Proposition 3.3 (Hyperelliptic theta constants and branch points). *Let Ω be a period matrix of a hyperelliptic curve*

$$\Sigma\colon Y^2 = X(X-1)\prod_{\ell=3}^{2g+1}(X - x_\ell). \tag{3.14}$$

Then for any $3 \leqslant \ell \leqslant 2g+1$, we have the following two types of formulae for branch points:

$$x_\ell(\Omega) = \tfrac{1}{2}\{1 + \theta_\ell(\Omega)\}, \tag{3.15}$$

where

$$\theta_\ell(\Omega) = \frac{\vartheta[\eta_{(W_1^{(\ell)}+1+\ell)\circ U}]^4 \cdot \vartheta[\eta_{(W_2^{(\ell)}+1+\ell)\circ U}]^4 - \vartheta[\eta_{(W_1^{(\ell)}+2+\ell)\circ U}]^4 \cdot \vartheta[\eta_{(W_2^{(\ell)}+2+\ell)\circ U}]^4}{\vartheta[\eta_{(W_1^{(\ell)}+1+2)\circ U}]^4 \cdot \vartheta[\eta_{(W_2^{(\ell)}+1+2)\circ U}]^4}.$$

Here $\vartheta[\eta_T]^4$ *means* $\vartheta[\eta_T](\Omega)^4$ *for any* T. *The reciprocals of roots can be expressed as follows*:

$$\frac{1}{x_\ell(\Omega)} = \tfrac{1}{2}\{1 + \theta^*_\ell(\Omega)\}, \tag{3.16}$$

where

$$\theta^*_\ell(\Omega) = \frac{\vartheta[\eta_{(W_1^{(\ell)}+1+2)\circ U}]^4 \cdot \vartheta[\eta_{(W_2^{(\ell)}+1+2)\circ U}]^4 - \vartheta[\eta_{(W_1^{(\ell)}+2+\ell)\circ U}]^4 \cdot \vartheta[\eta_{(W_2^{(\ell)}+2+\ell)\circ U}]^4}{\vartheta[\eta_{(W_1^{(\ell)}+1+\ell)\circ U}]^4 \cdot \vartheta[\eta_{(W_2^{(\ell)}+1+\ell)\circ U}]^4}.$$

Proof. For the first formula, let $k=1$ and $m=2$ in (3.13). For the second formula, let $k=1$, $\ell=2$ in (3.13), and then rewrite m as ℓ. $\square$

This formula expresses roots of a hyperelliptic curve (3.14) in terms of hyperelliptic theta constants with half integral characteristics associated to the period matrix Ω of the curve. Recall that to define Ω, we need to choose an ordering of branch points and a simple closed curve passing through branch points in the chosen order. A different choice of a simple closed curve leads to another period matrix Ω' related by the action of Γ_2. Since $\vartheta[\eta_S](\Omega)^4$'s are Siegel modular forms of weight 2 and of level Γ_2 by Proposition 3.2, the function $\theta_\ell(\Omega)$ is a Siegel modular function (of weight 0) of level Γ_2 on $\mathfrak{H}_g$. However, note that in (3.15) and (3.16), the value of $\theta_\ell(\Omega)$ is *finite* for hyperelliptic period matrix $\Omega \in \mathfrak{H}_g$. In fact, the denominator of $\theta_\ell(\Omega)$ does not vanish for hyperelliptic period matrix Ω [15, IIIa, Corollary 6.7].

4. Analytic hyperelliptic genera: Siegel modular function valued genera

In Section 2, we studied algebraic hyperelliptic genera whose values are in the polynomial algebra generated by indeterminates θ_k. In this section, we give analytic version of hyperelliptic genera by expressing θ_k's as weight 0 Siegel modular functions of level 2 on $\mathfrak{H}_g$. These Siegel modular functions have no poles along the locus of hyperelliptic period matrices.

Let Σ be a smooth hyperelliptic curve of genus g, and let $\{y_1, y_2, y_3, \ldots, y_{2g+1}, \infty\}$ be its marked branch points. We assume that none of y_i's are zero. Then Σ is isomorphic to a curve defined by a degree $2g+1$ polynomial:

$$\Sigma : Y^2 = F(X) = \prod_{\ell=1}^{2g+1}\left(1 - \frac{X}{y_\ell}\right). \tag{4.1}$$

We want to express coefficients of $F(X)$ as Siegel modular functions evaluated at hyperelliptic period matrix Ω of Σ. Since Ω is determined up to the action of Γ_2 for a chosen ordering of branch points, we expect that the resulting expression of coefficients of $F(X)$ is invariant under the action of Γ_2. In fact, this is indeed the case.

To apply Thomae's formula and the resulting formulae (3.15) and (3.16), we first consider a hyperelliptic curve Σ' isomorphic to Σ such that Σ' has ordered branch points $\{0, 1, x_3, \ldots, x_{2g+1}, \infty\}$. Let $T : \mathbb{C}P^1 \to \mathbb{C}P^1$ be an isomorphism such that $T(y_1) = 0$, $T(y_2) = 1$, $T(\infty) = \infty$. Such a map is linear and is given by $T(y) = (y - y_1)/(y_2 - y_1)$. Let $T(y_\ell) = x_\ell$ for $3 \leqslant \ell \leqslant 2g + 1$. Then

$$\Sigma' : Y^2 = X(X - 1) \prod_{\ell=3}^{2g+1} \left(1 - \frac{X}{x_\ell}\right).$$

The map T extends to an isomorphism $T : \Sigma \to \Sigma'$. By choosing a simple closed curve passing through branch points of Σ', we obtain a symplectic basis of $H_1(\Sigma'; \mathbb{Z})$ in a standard way. This then gives us a period matrix Ω for Σ'. By pulling back the symplectic basis of Σ' to Σ, we obtain the same period matrix Ω for Σ. By (3.15), we obtain an expression of x_ℓ's for $3 \leqslant \ell \leqslant 2g + 1$ as Siegel modular functions. Let $y_1 = \alpha - \beta$ and $y_2 = \alpha + \beta$, for some $\alpha, \beta \in \mathbb{C}$. Here, $\alpha \neq \beta$ since we are assuming that none of y_i's are zero. Then, $y_\ell = T^{-1}(x_\ell) = \alpha + \beta\theta_\ell(\Omega)$ for $3 \leqslant \ell \leqslant 2g + 1$. Thus, we have obtained an equation of Σ in terms of hyperelliptic period matrix Ω. This is part (I) of the next proposition. We have another representation of Σ in part (II) using different ordering of branch points.

Proposition 4.1. *Let Σ be a smooth hyperelliptic curve given by*

$$\Sigma : Y^2 = F(X) = \prod_{\ell=1}^{2g+1} \left(1 - \frac{X}{y_\ell}\right), \tag{4.2}$$

where none of y_i's are zero, and they are mutually distinct.

(I) *Let Ω be any period matrix associated to an ordering $\{y_1, y_2, y_3, \ldots, y_{2g+1}, \infty\}$ of branch points. Let $y_1 = \alpha - \beta$ and $y_2 = \alpha + \beta$ for some α, β. Then the coefficients of the hyperelliptic curve* (4.2) *can be expressed as Siegel modular functions of weight* 0 *and of level* 2 *as follows*:

$$\Sigma : Y^2 = \left(1 - \frac{X}{\alpha - \beta}\right)\left(1 - \frac{X}{\alpha + \beta}\right) \prod_{\ell=3}^{2g+1} \left(1 - \frac{X}{\alpha + \beta\theta_\ell(\Omega)}\right), \tag{4.3}$$

where $\theta_\ell(\Omega)$'s are as in (3.15). *Here, $W_1^{(\ell)}$ and $W_2^{(\ell)}$ are any subsets such that*

$$W_1^{(\ell)} \amalg W_2^{(\ell)} \amalg \{1, 2, \ell\} = \{1, 2, 3, \ldots, 2g + 1\} \tag{4.4}$$

with $\#W_1^{(\ell)} = \#W_1^{(\ell)} = g - 1$.

(II) *Let Ω' be any period matrix for an ordering $\{\infty, y_2, y_3, \ldots, y_{2g+1}, y_1\}$ of branch points. Let $1/y_1 = \alpha - \beta$ and $1/y_2 = \alpha + \beta$. Then in terms of the period matrix Ω', the coefficients of the hyperelliptic curve Σ in* (4.2) *can be represented as*

$$Y^2 = \{1 - (\alpha - \beta)X\}\{1 - (\alpha + \beta)X\} \prod_{\ell=3}^{2g+1} [1 - \{\alpha + \beta\theta_\ell^*(\Omega')\}X], \tag{4.5}$$

where $\theta_\ell^*(\Omega')$'s are as in (3.16), *and $W_1^{(\ell)}$ and $W_2^{(\ell)}$ are as in* (4.4).

Proof. We only have to prove part (II). Let $T:\mathbb{C}P^1 \to \mathbb{C}P^1$ be an isomorphism given by

$$T(y) = \frac{(y_1/y_2) - 1}{(y_1/y) - 1}.$$

Then $T(y_1) = \infty$, $T(y_2) = 1$, $T(\infty) = 0$. Letting $T(y_\ell) = z_\ell$ for $3 \leqslant \ell \leqslant 2g+1$, we have an order preserving bijection:

$$T: \{\infty, y_2, y_3, \ldots, y_{2g+1}, y_1\} \xrightarrow{\cong} \{0, 1, z_3, \ldots, z_{2g+1}, \infty\}.$$

Now we choose the above ordering of branch points for Σ in (4.2). Let Σ' be the curve $Y^2 = X(X-1)\prod_{\ell=3}^{2g+1}(X - z_\ell)$ with the above ordering of branch points. Let Ω' be any period matrix for Σ' corresponding to this ordering of branch points. Since T induces an isomorphism between Σ and Σ', we see that Ω' is also a period matrix for Σ with the above marked branch points. By Proposition 3.3, we have $1/z_\ell = \frac{1}{2}\{1 + \theta_\ell^*(\Omega')\}$ for $3 \leqslant \ell \leqslant 2g+1$. Letting $1/y_1 = \alpha - \beta$ and $1/y_2 = \alpha + \beta$, we have $1/y_\ell = \alpha + \beta\theta_\ell^*(\Omega')$ for $\ell \geqslant 3$. This proves the formula (4.5). □

Note that the expression of branch points y_ℓ of Σ in terms of hyperelliptic theta constants is invariant under the action of Γ_2 by Proposition 3.2, and it is independent of the choice of subsets $W_1^{(\ell)}$ and $W_2^{(\ell)}$.

By applying Theorem 2.2 to hyperelliptic integrals of the form $\int^X \mathrm{d}x/\sqrt{F(X)}$, where $F(X)$ is given by the right-hand sides of (4.3) and (4.5) with some constants $\alpha, \beta \in \mathbb{C}$, we obtain genera whose values are Siegel modular functions (of weight 0) and of level 2 (defined on the subspace $\mathfrak{H}_g^{\text{h.e.}}$ of hyperelliptic period matrices of genus g):

$$\varphi: \Omega_*^{\mathrm{U}} \to \mathscr{S}_0(\mathfrak{H}_g^{\text{h.e.}})^{\Gamma_2}. \tag{4.6}$$

Since the ring of Siegel modular functions (of weight 0) is an ungraded ring, the grading in Ω_*^{U} does not play a role. Note that α, β need not be constants. They can be some modular forms of level 2, in which case the genus has values in the ring of Siegel modular forms.

Also note that formulae (4.3) and (4.5) have the simplest form when $\alpha = 0$ and $\beta = 1$. In this case, the hyperelliptic curve is given by

$$Y^2 = (1 - X^2)\prod_{\ell=3}^{2g+1}\left(1 - \frac{X}{\theta_\ell(\Omega)}\right), \tag{4.7}$$

and

$$Y^2 = (1 - X^2)\prod_{\ell=3}^{2g+1}(1 - \theta_\ell^*(\Omega')X). \tag{4.8}$$

The hyperelliptic genera associated to hyperelliptic curves (4.7) and (4.8) admit the following description. We use (4.8) in our discussion. The result for (4.7) is similar.

Let

$$\prod_{g=3}^{2g+1}(1-\theta_\ell^*(\Omega)X)=1-\Theta_1X+\Theta_2X^2-\cdots-\Theta_{2g-1}X^{2g-1}, \tag{4.9}$$

where Θ_k is the kth elementary symmetric function in $\theta_\ell^*(\Omega)$'s. Recall that we defined elements $\lambda_n\in\Omega_{2n}^{\mathrm{U}}$ in (2.3) as polynomials in $[\mathbb{C}P^k]$'s with integral coefficients. Formula (2.11) gives another description in terms of Milnor manifolds. For $n\geqslant 0$, we let

$$\tilde{\lambda}_{2n}=1+\lambda_2+\lambda_4+\cdots+\lambda_{2n},\quad \tilde{\lambda}_{2n+1}=\lambda_1+\lambda_3+\cdots+\lambda_{2n+1}. \tag{4.10}$$

These are nonhomogeneous rational generators of $\Omega_*^{\mathrm{U}}\otimes\mathbb{Q}$.

Proposition 4.2. *Let $\varphi:\Omega_*^{\mathrm{U}}\to\mathscr{S}_0(\mathfrak{H}_g^{\mathrm{h.e.}})^{\Gamma_2}$ be the hyperelliptic genus of genus g whose logarithm is given by the formal integral*

$$g_\varphi(X)=\int_0^X\frac{\mathrm{d}X}{\sqrt{(1-X^2)\prod_{\ell=3}^{2g+1}(1-\theta_\ell^*(\Omega)X)}}. \tag{4.11}$$

Then the genus φ is described in terms of $\{\tilde{\lambda}_k\}_{k\geqslant 1}$ as follows:

$$\varphi(\tilde{\lambda}_k)=\begin{cases}(-1)^k\Theta_k & \text{if } 1\leqslant k\leqslant 2g-1,\\ 0 & \text{if } k\geqslant 2g.\end{cases} \tag{4.12}$$

Proof. From (4.9) and (4.11), we have $\{g_\varphi'(X)^2(1-X^2)\}^{-1}=1+\sum_{k=1}^{2g-1}(-1)^k\Theta_kX^k$. By (4.10),

$$\{g_{\mathrm{MU}}'(X)^2(1-X^2)\}^{-1}=\frac{\sum_{n\geqslant 0}\lambda_nX^n}{(1-X^2)}=\sum_{n\geqslant 0}\tilde{\lambda}_nX^n.$$

Since φ maps this power series to the polynomial above, we obtain our description (4.12). □

Next, we describe the genus 2 case explicitly. There are several reasons why we are interested in genus 2 case. This case is of most interest for us because every genus 2 Riemann surface is hyperelliptic, and the set of hyperelliptic period matrics is an open dense subset of the Siegel upper half space $\mathfrak{H}_2$ of genus 2 (both of complex dimension 3). Thus, our expression of branch points in terms of hyperelliptic theta constants are finite on an open dense subset of $\mathfrak{H}_2$. Also note that in genus 2 case, there are *no ambiguities* for the choices of $W_1^{(\ell)}$ and $W_2^{(\ell)}$ in Propositions 3.3 and 4.1, unlike higher genera cases. Thus, our formulae (4.2) and (4.5) for genus 2 case are in a sense "canonical" formulae for coefficients of hyperelliptic curves in terms of period matrices, up to choices of α, β.

Another reason is that the ring of genus 2 Siegel modular forms are fairly well understood [10–12] for various levels. For example, the ring of genus 2 Siegel modular forms of level 2 is described in [10, pp. 396–397, 406].

Yet another reason is a possible connection with genus 2 conformal field theory [21], where genus 2 partition functions are conjectured for some conformal field theories including Moonshine Module.

From (3.10), the genus 2 η-characteristics are given by

$$\eta_1=\begin{pmatrix}\frac12 & 0\\ 0 & 0\end{pmatrix},\quad \eta_2=\begin{pmatrix}\frac12 & 0\\ \frac12 & 0\end{pmatrix},\quad \eta_3=\begin{pmatrix}0 & \frac12\\ \frac12 & 0\end{pmatrix},$$

$$\eta_4=\begin{pmatrix}0 & \frac12\\ \frac12 & \frac12\end{pmatrix},\quad \eta_5=\begin{pmatrix}0 & 0\\ \frac12 & \frac12\end{pmatrix}. \tag{4.13}$$

Note that η_1,η_3,η_5 are even characteristices, and η_2,η_4 are odd characteristices. We use Part (II) of Proposition 4.1 to calculate coefficients of Eq. (4.2). Each theta constant in the expression of $\theta^*_\ell(\Omega)$ in (3.16) can be calculated easily using the following formula:

$$\vartheta\begin{bmatrix}\vec{a}+\vec{n}\\ \vec{b}+\vec{m}\end{bmatrix}(\Omega)=\exp(2\pi i\vec{a}\cdot\vec{m})\vartheta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\Omega).$$

By (4.5), coefficients of general smooth hyperelliptic curves whose branch points are all distinct and contains the point ∞ are given by

$$Y^2=(1-(\alpha-\beta)X)(1-(\alpha+\beta)X)\prod_{\ell=3}^{5}\{1-(\alpha+\beta\theta^*_\ell(\Omega))X\}, \tag{4.14}$$

$$\theta^*_3(\Omega)=\frac{\vartheta\begin{bmatrix}\frac12 & 0\\ 0 & 0\end{bmatrix}(\Omega)^4\cdot\vartheta\begin{bmatrix}\frac12 & \frac12\\ 0 & 0\end{bmatrix}(\Omega)^4-\vartheta\begin{bmatrix}0 & \frac12\\ \frac12 & 0\end{bmatrix}(\Omega)^4\cdot\vartheta\begin{bmatrix}0 & 0\\ \frac12 & 0\end{bmatrix}(\Omega)^4}{\vartheta\begin{bmatrix}0 & \frac12\\ 0 & 0\end{bmatrix}(\Omega)^4\cdot\vartheta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}(\Omega)^4},$$

$$\theta^*_4(\Omega)=\frac{\vartheta\begin{bmatrix}\frac12 & 0\\ 0 & \frac12\end{bmatrix}(\Omega)^4\cdot\vartheta\begin{bmatrix}\frac12 & \frac12\\ 0 & 0\end{bmatrix}(\Omega)^4-\vartheta\begin{bmatrix}0 & \frac12\\ \frac12 & 0\end{bmatrix}(\Omega)^4\cdot\vartheta\begin{bmatrix}0 & 0\\ \frac12 & \frac12\end{bmatrix}(\Omega)^4}{\vartheta\begin{bmatrix}0 & \frac12\\ 0 & 0\end{bmatrix}(\Omega)^4\cdot\vartheta\begin{bmatrix}0 & 0\\ 0 & \frac12\end{bmatrix}(\Omega)^4},$$

$$\theta^*_5(\Omega)=\frac{\vartheta\begin{bmatrix}\frac12 & 0\\ 0 & \frac12\end{bmatrix}(\Omega)^4\cdot\vartheta\begin{bmatrix}\frac12 & 0\\ 0 & 0\end{bmatrix}(\Omega)^4-\vartheta\begin{bmatrix}0 & 0\\ \frac12 & 0\end{bmatrix}(\Omega)^4\cdot\vartheta\begin{bmatrix}0 & 0\\ \frac12 & \frac12\end{bmatrix}(\Omega)^4}{\vartheta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}(\Omega)^4\cdot\vartheta\begin{bmatrix}0 & 0\\ 0 & \frac12\end{bmatrix}(\Omega)^4}.$$

Here, Ω runs over an open dense subset of the genus 2 Siegel upper half space $\mathfrak{H}_2$.

Observe that all the theta characteristics we see in (4.14) are *even*. There are 10 even characteristics for $g=2$ case among 16 possible half integral characteristics. In the above expressions, nine of the 10 even ones appear and they appear exactly twice, and the only remaining even theta characteristic is

$$\begin{bmatrix}\frac12 & \frac12\\ \frac12 & \frac12\end{bmatrix}.$$

Also we observe that if we let

$$\lambda_1 = \frac{\vartheta\begin{bmatrix}\frac12 & 0\\ 0 & 0\end{bmatrix}(\Omega)^4}{\vartheta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}(\Omega)^4}, \quad \mu_1 = \frac{\vartheta\begin{bmatrix}\frac12 & \frac12\\ 0 & 0\end{bmatrix}(\Omega)^4}{\vartheta\begin{bmatrix}0 & \frac12\\ 0 & 0\end{bmatrix}(\Omega)^4}, \quad \nu_1 = \frac{\vartheta\begin{bmatrix}\frac12 & 0\\ 0 & \frac12\end{bmatrix}(\Omega)^4}{\vartheta\begin{bmatrix}0 & 0\\ 0 & \frac12\end{bmatrix}(\Omega)^4},$$

$$\lambda_2 = i\frac{\vartheta\begin{bmatrix}0 & 0\\ \frac12 & 0\end{bmatrix}(\Omega)^4}{\vartheta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}(\Omega)^4}, \quad \mu_2 = i\frac{\vartheta\begin{bmatrix}0 & \frac12\\ \frac12 & 0\end{bmatrix}(\Omega)^4}{\vartheta\begin{bmatrix}0 & \frac12\\ 0 & 0\end{bmatrix}(\Omega)^4}, \quad \nu_2 = i\frac{\vartheta\begin{bmatrix}0 & 0\\ \frac12 & \frac12\end{bmatrix}(\Omega)^4}{\vartheta\begin{bmatrix}0 & 0\\ 0 & \frac12\end{bmatrix}(\Omega)^4},$$

then $\theta_\ell^*(\Omega)$'s in (4.14) can be written as

$$\theta_3^*(\Omega) = \lambda_1\mu_1 + \lambda_2\mu_2, \quad \theta_4^*(\Omega) = \mu_1\nu_1 + \mu_2\nu_2, \quad \theta_5^*(\Omega) = \nu_1\lambda_1 + \nu_2\lambda_2,$$

exhibiting some symmetries among $\theta_\ell^*(\Omega)$'s. The hyperelliptic curve (4.14) with some constants $\alpha, \beta \in \mathbb{C}$ gives rise to a hyperelliptic genus

$$\varphi : \Omega_*^{\mathrm{U}} \to \mathscr{S}_0(\mathfrak{H}_2)^{\Gamma_2}$$

with values in the field of Γ_2-invariant Siegel modular functions of weight 0 on $\mathfrak{H}_2$.

By changing the choice of α, β, we can get a hyperelliptic genus with values in the ring of Siegel modular forms of level 2 with positive weight. For example, let $\alpha = 0$ and

$$\beta = \vartheta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}(\Omega)^4 \cdot \vartheta\begin{bmatrix}0 & \frac12\\ 0 & 0\end{bmatrix}(\Omega)^4 \cdot \vartheta\begin{bmatrix}0 & 0\\ 0 & \frac12\end{bmatrix}(\Omega)^4.$$

Then, the roots of (4.14) can be written as

$$\beta\theta_3^*(\Omega) = \vartheta\begin{bmatrix}\frac12 & 0\\ 0 & 0\end{bmatrix}^4 \cdot \vartheta\begin{bmatrix}0 & 0\\ 0 & \frac12\end{bmatrix}^4 \cdot \vartheta\begin{bmatrix}\frac12 & \frac12\\ 0 & 0\end{bmatrix}^4 - \vartheta\begin{bmatrix}0 & 0\\ \frac12 & 0\end{bmatrix}^4 \cdot \vartheta\begin{bmatrix}0 & 0\\ 0 & \frac12\end{bmatrix}^4 \cdot \vartheta\begin{bmatrix}0 & \frac12\\ \frac12 & 0\end{bmatrix}^4,$$

$$\beta\theta_4^*(\Omega) = \vartheta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}^4 \cdot \vartheta\begin{bmatrix}\frac12 & 0\\ 0 & \frac12\end{bmatrix}^4 \cdot \vartheta\begin{bmatrix}\frac12 & \frac12\\ 0 & 0\end{bmatrix}^4 - \vartheta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}^4 \cdot \vartheta\begin{bmatrix}0 & \frac12\\ \frac12 & 0\end{bmatrix}^4 \cdot \vartheta\begin{bmatrix}0 & 0\\ \frac12 & \frac12\end{bmatrix}^4,$$

$$\beta\theta_5^*(\Omega) = \vartheta\begin{bmatrix}\frac12 & 0\\ 0 & 0\end{bmatrix}^4 \cdot \vartheta\begin{bmatrix}0 & \frac12\\ 0 & 0\end{bmatrix}^4 \cdot \vartheta\begin{bmatrix}\frac12 & 0\\ 0 & \frac12\end{bmatrix}^4 - \vartheta\begin{bmatrix}0 & \frac12\\ 0 & 0\end{bmatrix}^4 \cdot \vartheta\begin{bmatrix}0 & 0\\ \frac12 & 0\end{bmatrix}^4 \cdot \vartheta\begin{bmatrix}0 & 0\\ \frac12 & \frac12\end{bmatrix}^4,$$

and they are all Siegel modular forms of weight 6 and of level 2, including β. Consequently, the hyperelliptic integral

$$\int^X \frac{\mathrm{d}X}{\sqrt{(1-\beta^2X^2)(1-\beta\theta_3^*(\Omega)X)(1-\beta\theta_4^*(\Omega)X)(1-\beta\theta_5^*(\Omega)X)}}$$

defines a hyperelliptic genus $\varphi : \Omega_*^{\mathrm{U}} \to \mathscr{S}_{6*}(\mathfrak{H}_2)_{\Gamma_2}$ with values in the ring of Siegel modular forms of level 2 whose weights are multiples of 6. This level 2 ring is well understood [10].

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